AN INVESTIGATION TO DETERMINE THE MASTERY OF SPECIFIC MATHEMATICAL UNDERSTANDING BY JUNIOR HIGH SCHOOL MATHEMATICS STUDENTS

A Thesis
Presented to
the Faculty of the School of Education
Boston University

In Partial Fulfillment
of the Requirements for the Degree
Master of Education

by
Arnold G. Adams

July 1949

First Reader: Henry W. Syer, Assistant Professor of Education

Second Reader: Robert L. Burch, Assistant Professor of Education

Third Reader: William C. Kvaraceus, Professor of Education

TABLE OF CONTENTS

INTRODUCTION

Educational and psychological research has contributed greatly to more effective teaching. There have been many studies showing that understanding facilitates learning and that transfer of training is increased by better understanding during the learning process. As a result of many research studies, in this field, educators are generally agreed that one of the major objectives in the teaching process is the development of understandings. Agreement in educational practice is not as common as agreement in the importance of understandings as educational objectives. This lag in practice is due to the following causes: 1, teachers were not conscious of the need to develop understanding; 2, teachers lacked themselves the knowledge of understanding; 3, teachers found it difficult to teach for understanding; 4, teachers found it difficult to evaluate growth in understanding.

CHAPTER I

AN ANALYSIS OF THE SPECIFIC BASIC MATHEMATICAL UNDERSTANDINGS OF THIS INVESTIGATION

Statement of the problem. The purpose of this study is to determine to what extent specific basic mathematical understandings taught in grades one through six are mastered by students in grades seven through nine in courses of mathematics in the public schools of Milton, Massachusetts.

Basic mathematical understanding is defined in this investigation as the intellectual power or the capacity of the pupil to form reasoned judgments that rise in the computational processes taught in grades one through six. Mastery of these basic mathematical understandings, for the purpose of this study, shall be considered as having been attained if the student is able to identify the correct response with the given illustration.

The teaching and measurement of understandings are most important educational objectives as will be noted in this chapter, but little has been done in most schools toward teaching for understanding and still less in the

measurement of any understandings that are taught. This vital lack of a measuring instrument became apparent during the search for a test to carry out the purpose of this study. With no adequate test available it was necessary to construct such a device for measuring understanding.

Method of investigation. The method used to determine the extent of the mastery of the specific basic mathematical understandings is through the administering of a Test of Basic Mathematical Understandings. Test items included in the instrument measured only mathematical understandings that are basic to computational processes taught in grades one through six. In order to eliminate the effect of mechanical manipulation inherent in computation, the test items were constructed in such manner as to require no computing. The results of this test will form the basis upon which the conclusions and implications of the study are made. The word "mathematics" will be used to include "arithmetic" in the study. The Forty-Fifth Yearbook[1] of the National Society for the Study of

---

[1]National Society for the Study of Education. The Measurement of Understanding, 45th Yearbook, Bloomington, Illinois. Public School Publishing Company, 1946, p. 138.

Education advocates the adoption of the word "mathematics" in preference to "arithmetic". In their words:

> During the past decade an increasing number of schools have been using the term elementary school mathematics to replace arithmetic in courses of study and in school reports. This change is no mere whimsey. It is indicative of a corresponding broadening of our vision of the content and function of arithmetic or mathematics in the elementary school. To many school people the word "arithmetic" was synonymous with computation; and arithmetic was merely a tool to be called forth when a need was recognized. More recent literature has called attention to the breadth of aims that should be achieved. For example, in one discussion of curriculum problems in the field, the following classification of aims is employed; (a) concepts and vocabulary, (b) principles and relationships, (c) social and economic information, (d) factual information, (e) processes and manipulations, (f) problems and basic thought patterns, (g) reflections and judgments.

Limitations of the study. This investigation is not an all inclusive criticism or justification of teaching results in mathematics in the Milton schools but is essentially a survey of existing conditions. The test of mathematical understandings used in this study is built around the topics considered by the National Council of Teachers of Mathematics[1] to be the core of the computa-

---

[1]National Council of Teachers of Mathematics. "The Place of Mathematics in Secondary Education", 15th Yearbook. Bureau of Publications, Teachers College, Columbia University, 1940.

tional work of the grades one through six, and are listed as objectives in the elementary school curriculum of Milton.

The core topics are:

1. Place Value and the decimal system.

2. The four fundamental processes with whole numbers.

3. The four fundamental processes with fractions.

4. The four fundamental processes with decimals.

The fifty basic mathematical understandings measured are listed in Appendix A.

Assumptions of the study. The following assumptions are made in the belief that they are reasonable and that they are necessary in making any significant interpretation of the results of the investigation.

1. It is assumed that the development of mathematical understandings basic to place value, the four fundamental processes with whole numbers, fractions, decimals is an acceptable part of arithmetic instruction in the elementary grades.

2. It is assumed that further development and maintenance of these understandings is an acceptable part of the arithmetic program in grades seven, eight and nine.

3. It is assumed that the test of basic mathematics used as a basis for the investigation is composed of items which really are basic to the mathematical understandings and are valid objectives for junior high school pupils.

4. It is assumed that a correct response to an item on the test implies mastery of the understanding involved and that an incorrect response to an item implies lack of mastery of the understanding involved.

5. It is assumed that incorrect responses to test items are due to lack of mastery of the understanding involved and not to lack of clearness or simplicity in the statement of the test items.

Importance of the problem and justification for the study. Three questions keep recurring as we examine the content of this study. (1) Should we teach for mathematical understandings in the computational aspects of mathematics? (2) Is it sufficient to teach only for speed and accuracy in the computational aspects of mathematics? (3) If we find that we should teach for mathematical understanding in addition to speed and accuracy in the computational aspects of mathematics, to what extent is this being done?

Of these three questions, we are concerned with only the first as this is an investigation to determine the mastery of specific basic mathematical understandings rather than the extent of mastery of mathematical computational skills and abilities. The need for the development of these higher mental abilities in mathematics (understandings, concepts, principles, generalizations, relationships) has come to be regarded as one of primary importance.

A closer examination of the trends in teaching mathematics shows that the problem of teaching for more than skill in computational techniques is by no means a new one. From ancient times the word arithmetic implied the theory of numbers and of number systems and only recently has it lost most of this meaning on elementary levels, and acquired instead the meaning of skill or facility in manipulating numbers. The Greeks and Romans used the word <u>arithmetic</u> to refer to the science of numbers -- <u>numerorum scientia</u>; and used the word <u>logistic</u> to refer to the more humble type of learning, manipulating numbers or computing.

Prior to 1821 the deductive method of teaching mathematics was used. During this period the treatment of arithmetic as a science of numbers decreased as its use as a tool of business men increased. Meaning and understanding were subordinated to speed and accuracy. Students were given a rule governing the manipulations of the process, made to memorize the rule, and then shown applications of the rule. This marks the beginning point in the trend away from arithmetic as a science of numbers to arithmetic as a series of unrelated computational skills.

Arithmetic was taught by the deductive method until 1821. At that time there was published the first textbook based upon the inductive method of reasoning. Warren Colburn was the author of this volume. In a later book entitled "Intellectual Arithmetic Upon the Inductive Method"[1] Colburn presents his case for the development of understandings, principles and generalizations.

---

[1]Colburn, Warren, Intellectual Arithmetic Upon the Inductive Method, Concord:Sanborn and Company, 1840, p. IV.

The idea of number is first acquired by observing sensible objects. Having observed that this quality is common to all things with which we are acquainted, we obtain an abstract idea of number. We first make calculations about sensible objects; and we soon observe that the same calculations will apply to things very dissimilar; and finally, that they may be made without reference to any particular things. Hence from particulars we establish general principles which serve as the basis of our reasonings, and enable us to proceed, step by step, from the most simple to the most complex operations. It appears that mathematical reasoning proceeds as much on the principle of analytic induction, as that of any other science.

Examples of any kind upon abstract numbers are of very little use, until the learner has discovered the principle from practical examples. They are more difficult in themselves for the learner does not see their use, and therefore does not so readily understand the question. But questions of a practical kind, if judiciously chosen, show at once what the combination is and what is to be effected by it. Hence the pupil will much more readily discover the means by which the result is to be obtained. The mind is also greatly assisted in the operations by reference to sensible objects. When the pupil learns a new combination by means of abstract examples, it very seldom happens that he understands practical examples more easily for it, because he does not discover the connexion until he has performed several practical examples, and begins to generalize them.

After the pupil comprehends an operation, abstract examples are useful to him, and make him familiar with it. And they serve better to fix the principle, because they teach the learner to generalize.

As has been previously stated, the concern on the part of students in the field for greater emphasis on the development of mathematical understanding has increased greatly in the past few years. The evidence of this concern is the increased frequency of published articles

and studies. Several of these articles will now be examined.

The Harvard Report[1] brought the problem into print with the following words:

> Mathematics comprises both abstraction and the application of the results obtained by abstraction to specific real problems. Of these aspects, the basic one is abstraction. Only because it is abstract is mathematics applicable generally to problems which arise in widely different areas.

Concerning the development of understandings on the part of "less gifted" pupils the Harvard report says in part:

". . . it is, of course, desirable to stimulate the interest of mathematically inept students in the number relations of arithmetic . . ."

---

[1]General Education in a Free Society: Report of the Harvard Committee, Cambridge, Harvard University Press, Cambridge, Massachusetts, 1945.

The Joint Commission of the Mathematics Association of America and the National Council of Teachers of Mathematics[1] have this to say regarding the mathematics program of the elementary school:

> The mathematics program of the elementary school is the indispensable foundation of all the pupil's later mathematics work. If that foundation is weak, the pupil's subsequent progress is likely to be permanently handicapped. It is assumed that a pupil who is adequately prepared for the work of the seventh grade has acquired a working knowledge of the arithmetic commonly taught in the primary schools. Hence the following attainments may be regarded as the normal equipment of the American pupil who has satisfactorily completed the work of the sixth grade.
>
> (1) A familiarity with the basic concepts, processes and the vocabulary of arithmetic.
>
> (2) Understandings of the significance of the different positions that a given digit may occupy in a number, including the case of the decimal fraction.

It should be noted that the first two attainments listed by the Commission are concerned with concepts and understandings.

---

[1]National Council of Teachers of Mathematics: The Place of Mathematics in Secondary Education, 15th Yearbook, New York: Bureau of Publications, Teachers College, Columbia University, 1940.

Later on in the report in a section entitled "Essentials of a General Program in Secondary Mathematics", the Commission again calls attention to the need for emphasis on the development of principles and understandings in the teaching of arithmetic.

> There should be a growing familiarity with the basic vocabulary and working principles of arithmetic. This involves, (1) giving an example or an informal explanation of the meaning of given terms, and, at a higher level, (2) developing formal definitions of terms that have a broad operational significance.
>
> In all teaching of secondary mathematics much attention should be given to a conscious grasp of the principles which underlie the fundamental processes of arithmetic. Examples of such principles are the following:
>
> (1) The numerator and denominator of a common fraction may be multiplied or divided by the same (non-zero) number, without changing the value of the fraction.
>
> (2) The order of the factors in a product does not affect the result.

Judd in his Psychological Analysis of the Fundamentals of Arithmetic[1] says:

> General ideas are the most important products of instruction in arithmetic. The fundamentals of arithmetic are general ideas and general formulas, not a multitude of special skills.

---

[1] Judd, Charles H., Psychological Analysis of the Fundamentals of Arithmetic, Monograph #32, Department of Education, University of Chicago, 1927.

Again in his Education as Cultivation of the Higher Mental Processes[1] Judd says:

> The chief difficulties encountered by present day teaching of arithmetic arises from the fact that schools, at least the better schools, are attempting to develop in pupils an understanding of number rather than merely drilling them in the use of tables and formal rules. When the effort is made to develop understandings, education is aiming at the cultivation of higher mental processes.

Buswell[2] speaking on weakness in present day arithmetic programs says:

> The emphasis on the concrete as a necessary proceding to the abstract has brought great gains to arithmetic. However, along with this signal gain in meaning there has come a tendency to stop with the concrete and never to arrive at significant abstractions which are really the essence of arithmetic.
>
> An abstraction is a generalization that grows out of concrete experience. The value of abstractions in arithmetic is due to their great convenience in short-circuiting the awkward process of using concretes in our thinking. At heart arithmetic is a system of abstract relationships which may give a far higher proficiency to number experiences than will ever be possible by the simpler and cruder processes of concrete reasoning.

---

[1]Judd, Charles H., Education as Cultivation of the Higher Mental Processes, New York:MacMillan Company, 1936.

[2]Buswell, Guy T., School Science and Mathematics, March, 1943, 43:201-12.

> It is just as important that arithmetic carry through to the point of abstraction as that it begin with the point of concreteness. A failure to carry through leaves pupils without the essential tool which arithmetic can contribute to their quantitative thinking.

Still further evidence of the concern for greater emphasis on the development of principles, understandings and generalizations in the teaching of mathematics comes from Brownell, Buckingham and Betz.

Brownell[1] says:

> If teachers took the vow to teach no arithmetic idea, process, or skill unless they could make it sensible to children, they would have to change drastically their classroom practices. They would find that they have to teach arithmetic mathematically for the sense in arithmetic inheres in the mathematics of numbers and of number processes. This statement does not mean that visual and other sensory aids would be discontinued. Quite the contrary, their use would be doubled or trebled for through such aids many mathematical meanings and relationships are most readily represented.

Buckingham[2] says:

> The task of the school is too exclusively understood to be that manipulation of symbols which we call computing. The school task in respect to arithmetic, is in reality far more fundamental. It is nothing less than an attack

---

[1] Brownell, William A., "Essential Mathematics for Minimum Army Needs, School Review, 52:484-92, October, 1944.

[2] Buckingham, B. R., "The Contribution of Arithmetic to a Liberal Education", The Mathematics Teacher, 35:147-160.

upon arithmetic illiteracy. There is an illiteracy applicable to the quantitative ideas just as there is an illiteracy applicable to the generalizations and concepts expressed in words instead of figures. In each case competence or illiteracy, is something more than the manipulation of the symbols. It is an appreciation of the meaning attached to the symbols and an ability to apply the symbols in order to facilitate thought.

Betz[1] says:

Mathematics is not merely a matter of computation, of juggling a few formulas. Of even greater importance are the generalized habits which are characteristic of mathematics together with its types of comprehension and appreciation and its modes of thinking. In fact, it is these elements primarily which are the carriers of the "functional" values of mathematics. And it is this broader concept of mathematics which underlies our entire material civilization, our age of science and industry.

Again Brownell[2] says:

One of the great fallacies of the elementary curriculum is to classify arithmetic as a skill or a drill or a tool subject. When arithmetis is viewed in these terms and is taught accordingly, the results are just what we have been getting for the last several decades; in a word, arithmetical incompetence. The teaching process, according to the tool conception of arithmetic, undertakes to tell children what to do

---

[1]Betz, William, "The Necessary Redirection of Mathematics; Including Its Relation to National Defense", The Mathematics Teacher, 35:147-160, April, 1942.

[2]Brownell, William A., "When Is Arithmetic Meaningful?", Journal of Educational Research, 38:481-98, March, 1945.

> (but not why to do it) and then by ceaseless drill to have them do it until they can demonstrate some degree of mastery. After that, heavy programs of maintenance are organized to keep the skills alive. But arithmetic, properly conceived, isnnot a tool or a drill subject. Of course, proficiency is necessary, everyone agrees that this is so; but more than proficiency (speed-correctness) in computation is demanded by the conditions of life. In practical living we must be intelligent in quantitative situations. Mechanical skills may suffice so long as these skills are employed in situations which are wholly familiar. To the degree that situations differ from the completely familiar, we must be able to think and one does not think effectively with mechanical skills alone. For many years we have been told that skills can be used intelligently only when they are acquired intelligently; hence the importance of meanings in arithmetic.

It would be erroneous to assume that the foregoing opinions in support of teaching for generalizations or understandings are representative of the thinking of all teachers of mathematics. On the contrary the opinion of Buell[1] on the subject may be representative of a sizable portion of the opposition group.

---

[1]Buell, Irwin A., "Let Us Be Sensible About It", The Mathematics Teacher, 37:306-8, November, 1944.

> One does not have to know the why of everything in order to enjoy life and to add to human happiness, and this is as true in mathematics as it is in other fields. One can use the telephone without knowing the basic theory of operation. Or one can know the basic theory and yet not have it in mind every time one answers its ring. Most people will never know the theory; nor does that matter much. Then why must every child be taught so many underlying reasons in the field of mathematics? He does not need to know the scientific reason back of everything he does so long as that which is done is done correctly.
>
> Article after article has been printed in which the author has said that the pupil in mathematics must consciously understand each time he performs a definite operation just the reason why he is permitted to do as he does. I should like to take exception to such statements. Since the mathematical ability of many is low and the total amount they will learn is limited, it is best to reduce many things to routine so that they may go on farther with practical work than they could go if we insisted on "completeness" all along the way.
>
> 'The keynote of the new arithmetic is that it should be meaningful rather than mechanical'. I say let's make it increasingly mechanical and then go on to something more abstract. Let us continually make the difficult into the mechanical and go on to the more difficult.

Although this point of view is held by many teachers, it is hardly a defensible one. In this article, Buell likens the teaching of understandings and principles in arithmetic to the teaching of arbitrary associations in spelling. He later extends his thinking to state that the methods of teaching used in bringing about achievement in spelling, namely; drill, repetition, can also be used in

the teaching of understandings, principles and generalizations.

When he advocates making arithmetic increasingly mechanical, Buell reflects a lack of knowledge of the findings of many research studies in the field of the psychology of arithmetic. Taking the meaning out of any learning process can hardly increase the rate of learning, nor can it decrease retroaction.

SUMMARY OF CHAPTER I

The foregoing references represent the opinions of students concerning the major purpose and emphasis in the teaching of elementary school mathematics. The division in thinking is clear-cut. On one hand we have the group advocating a theory of teaching the computational aspects of mathematics that is based upon the development of understandings inherent in the relations of numbers. This is known as the meaning theory.

On the other hand we have the group that advocates a theory for teaching the computational aspects of mathematics that is based upon learning through repetition of the mechanical manipulations. This is better known as the drill theory.

CHAPTER II

PSYCHOLOGIES AND LEARNING THEORIES IN MATHEMATICS

A study of the literature concerned with psychologies and learning theories in arithmetic reveals a very marked trend during the past few years. This trend represents a shift from the "connectionist" or "drill" theory to the "generalization" or "meaning" theory as a basis for instruction in arithmetic. The meaning theory is not the new theory that many educators believe it to be but is actually the older of the two. The proof of this has already been offered in the previous chapter. The name arithmetic itself has only recently acquired the meaning of skill or facility in manipulating numbers --- *numerorum scientia*. It is only within the last few centuries that this original meaning has been altered.

The shift away from the drill theory as a method of teaching arithmetic began with the inductive plan advocated by Warren Colburn. Dewey and McLellan[1] in 1895 published *The Psychology of Number* in which they criticize the drill theory.

---

[1]McLellan, J. A. and Dewey, J., *The Psychology of Number*, D. Appleton and Company, 1895, p. 85.

> The method which considers number simply as a plurality of fixed units, necessarily leads to exhausting and meaningless mechanical drill. . . . no one can deny that, however much it is sought to add interest to this study (by the introduction of various objects, counting eyes, ears, etc., dividing the children into groups etc.) the process is essentially one of mechanical drill. The interest afforded by the objects remains, after all, external and adventitious to the numbers themselves. Moreover, the appeal is constantly made simple to the memorizing power.

Buckingham[1] says:

> . . . many of us were brought up on an atomistic psychology --- a psychology in virtue of which we divided and subdivided arithmetical processes into types and unit skills and undertook to teach each minute part in the conscientious belief that only so could the process as a whole be mastered. This atomistic psychology is giving away to one which emphasizes not the parts but the whole --- in short, to an organismic psychology.

McConnell[2] states that the trend in the psychology of learning emphasizes the "primacy of organization" and that this trend is reflected in the research being conducted in this field. The meaning theory emphasizes relatedness rather than atomization and itemization. It stresses generalization rather than specificity and considers the development of understandings more important than the development of skill through repetition. Under the drill

---

[1] Buckingham, B. R., "Significance, Meaning and Insight, These Three", Mathematics Teacher, 31:24-30, January, 1938.

[2] McConnell, T. R., National Council of Teachers of Mathematics, 16th Yearbook. Arithmetic in General Education. Chapter XI Recent Trends in Learning Theory. pp. 268-289.

theory, learning is a process of stamping in; under the meaning theory, learning is a developmental process. The latter method encourages the use of problem situations and the discovery of relationships as opposed to parrot like repetition.

Dickey[1] says:

> The transformation to a new frame of reference in the objectives of arithmetic has been concomitant with the newer psychological explanation of learning, which has important significance for both the content and the method of arithmetic.

The drill theory as a method of teaching arithmetic. The drill theory has been the most widely used method for the past fifty years in the teaching of arithmetic. Brownell[2] gives as evidence of this popularity the widespread use of workbooks and other forms of seatwork materials, and the great concern on the part of the teacher, supervisor and administrator for speed and accuracy in computation.

---

[1]Dickey, J. W., "Arithmetic and Gestalt Psychology", Elementary School Journal, 39:46-53, Sept., 1938.

[2]Brownell, W. A., National Council of Teachers of Mathematics. The Teaching of Arithmetic, 10th Yearbook, Chapter I, "Psychological Consideration in the Learning and Teaching of Arithmetic." New York Bureau of Publications. Teachers College, Columbia University, 1935.

The organization of the arithmetic textbooks are additional evidence of the popularity of the drill theory. An examination of any text will show an increasing amount of drill material. The textbooks in arithmetic have changed from a single volume used in 1900 in all grades to our present book series of one for each grade.

Brownell sums up the popularity of the drill theory by saying that it is "due to two misleading approaches to a definition of arithmetic ability: (a) analysis of adult uses of arithmetic, (b) the 'bond' theory of learning."

The drill theory defined. Arithmetic, according to the drill theory, is composed of a large number of unrelated facts and independent skills and abilities. The method of learning is essentially repetition, with a premium placed on speed and accuracy of response. The chief learning activity is devoted to watching the teacher perform the operation and then the children perform the operation themselves. The teacher spends little or no time developing understandings of the processes.

We again look to Brownell[1] to summarize the main

---

[1]Brownell, W. A., op. cit., p. 2.

points in the teaching procedure based on the drill theory:

1. Arithmetic for the purpose of learning and teaching, may be analyzed into a great many units or elements of knowledge and skill which are comparatively separate and unconnected.

2. The pupil is to learn these almost innumerable elements whether he understands them or not.

3. The pupil is to learn these elements in the form in which he will subsequently use them.

4. The pupil will attain these ends most economically and most completely through formal repetition.

Nature and support of the drill theory. One of the chief exponents of connectionist psychology is Edward L. Thorndike[1]. The following is what he writes concerning the application of this psychology to arithmetic. In his Psychology of Arithmetic he says:

> The psychology of the elementary school subjects is concerned with the connections whereby a child is able to respond to the sight of printed words by thoughts of their meanings, to the thought of 'six and eight' by thinking 'fourteen'. The aims of elementary education, when fully defined, will be found to be the production of changes in human nature represented by an almost countless list of connections or bonds whereby the pupil thinks or feels or acts in certain ways in response to the situation the school has organized.
>
> The psychology of the school subjects tries to analyze it into constituent bonds, to decide what bonds need to be formed and in what order as means to the most economical attainment of the desired improvement.

---

[1]Thorndike, Edward L., Psychology of Arithmetic, New York:MacMillan Company, 1932, pp. xi-xii.

Thorndike[1] modified his thinking in later writings but it was this interpretation of the drill theory which has most widely influenced the teaching of arithmetic. These influences on the courses of study and the teaching methods of arithmetic will be noted now.

The teaching of arithmetic that is based upon this psychology is mainly concerned with two tasks:

1. What bonds are to be fixed?

2. What practice or drill will fix these bonds?

The appeal in this psychology was its simplicity. The influence on courses of study and teaching methods has been tremendous. Evidence of this influence on teaching methods is noted when Thorndike reports in table form the opinions of teachers as to the amount of practice they believe is required for each of the basix addition facts and subtraction facts in books I and II of the average three book text in arithmetic. The group included fifty experienced teachers. For the arithmetical fact 3 plus 2, the median estimate was 1500 and the highest estimate was one million! This table represents the extent to which some teachers were influenced by a psychology that advocated the need for repeating a given fact many times before it is fixed.

---

[1]Thorndike, E. L., op. cit., p. 124.

Morton[1] advances support of the drill theory when he discusses, "Advances in Educational Psychology".

> Again, we understand better how habits are formed, how they function, and how practice on the fundamental bonds should be distributed. Arithmetic processes have been analyzed and the details teased out where they can be seen. We train pupils in all the bonds involved in higher decade addition up to a set limit, say to 39.

Brueckner[2] emphasizes the analytical treatment as far as the elements involved in computation in arithmetic are concerned. Arithmetic is made up of a hierarchy of habits, specific skills and general abilities. Each may be isolated, studied independently, and have its elements determined by critical analysis. This fact has long been recognized by those who have attempted to evaluate the results of instruction by means of educational tests and to adapt the instruction to the needs and capacities of pupils as revealed by these tests.

---

[1]Morton, R. L., Teaching Arithmetic in the Intermediate Grades. New York:Silver Burdett and Company, 1927, p. 5.

[2]Brueckner, L. J., Diagnostic and Remedial Teaching in Arithmetic. Philadelphia:J. C. Winston Company, 1930, p. 13.

Reed[1] stresses the importance of drill in the learning of arithmetic.

"Arithmetic lends itself readily to training by repetition."

Concerning the importance of habit formation, Thorndike[2] offers the following:

> The importance of habit formation or connection making has been grossly underestimated by the majority of teachers and writers of textbooks. Children as a rule do not deduce their method of manipulation from their knowledge of decimal notation. They learn the method of manipulating numbers by more or less blindly acquiring them as associate habits.

A teacher's understanding of the psychology underlying a given text is to a considerable extent derived from the manual or guide accompanying the text. The following statement taken from such a manual[3] might be a strong influence in guiding a teacher.

---

[1]Reed, H. B., Psychology of Elementary School Subjects, Boston:Ginn and Company, 1927, p. 175.

[2]Thorndike, E. L., op. cit., pp. 70-71

[3]Wheeler, L. R., "A Comparative Study of the Difficulty of the 100 Addition Combinations", Journal of Genetic Psychology, 54:205-312, June, 1939.

> The end sought in manipulating the addition and subtraction combinations is automatic response. It therefore follows that the teaching technique which most nearly eliminates any dependence upon reasoning or any other clumsy method, is the best.

Thorndike[1] says the best way to develop an understanding of arithmetic is to learn to operate by imitation and the extension of past knowledge; then to make sure that the operation is right by verification from known facts and last of all to learn why it is right and must be right.

Criticism of the drill theory. Criticisms of a theory of arithmetic teaching which places emphasis on repetition and mechanical manipulation of figures is long standing. McLellan and Dewey[2], writing over fifty years ago, challenged the method by saying that it is little more than blind manipulation of symbols. The child takes the figures and performs certain tricks which are dignified by the name addition, subtraction, multiplication, etc. He knows very little of what the figures signify and less of the meaning of the operations.

---

[1]Thorndike, E. L., op. cit., p. 212.

[2]McLellan, J. A. and Dewey, J., op. cit., p. 60.

Dewey[1] holds the same viewpoint thirty years later when he writes:

> In some educational dogmas and practices, the very idea of training the mind seems to be hopelessly confused with that of drill which hardly touches mind at all, or touches it for the worse, since it is wholly taken up with training skill in external execution. This method reduces the training of human beings to the level of animal training. Practical skill, modes of effective technique, can be intelligently, non-mechanically used, only when intelligence has played a part in their acquisition.

Hullfish[2] criticizes Thorndike's thesis that all learning is analytic on the grounds that the thesis has no "internal consistency". He further contends that the thesis carries with it a false connotation, namely, that learning is all analysis. This thesis excludes the role of synthesis in the learning process, and the interaction of the two processes. Any attempt to separate the two processes and advocate the one to the exclusion of the other renders harm to the nature of the learning process.

---

[1]Dewey, J., How We Think, Boston:D.C. Heath and Company, 1933, p. 52.

[2]Hullfish, H. G., "Aspects of Thorndike's Psychology in Their Relation to Educational Theories and Practices", Ohio State University Studies #1, Columbus, Ohio, 1926, p.69.

Bailey[1] offers nearly the same criticism of Thorndike's influence on teaching methods. By adhering to a narrow teaching pattern based on repetitive activities, reasoning is subordinated. Instead of increasing the efficacy of the learning, the drill theory tends to lower it. Teachers tend to circumvent the intellectual activity connected with the act of learning, and tend to replace it with memorization of rules, following of steps or procedures, and exposition of model solutions. He further asserts that this type of teaching brings about such a low level type of learning that pupils on leaving school are "almost helpless to think out by themselves the solution of problems that depart from type".

Shift in thinking from drill theory to meaning theory. Because of so many criticisms and the supporting evidence of research studies carried out by psychologists in many learning situations, there came about a shift in thinking in the minds of educators connected with this problem. Both psychologists and educators began to doubt the soundness of a psychology which stressed repetition as a method of learning. Writers in the field reflected this trend in their work.

---

[1]M. A. Bailey, "The Thorndike Philosophy of Teaching the Processes and Principles of Arithmetic", Mathematics Teacher, 16:129-140, March, 1923.

Suzzalo[1] in 1911 gives recognition of the new development but still advocates drill as the best method of teaching arithmetic. He directs our attention to the fact that objective instruction is the first step in teaching based on the inductive method. In another paragraph, however, he cautions teachers against a tendency to rationalize the processes in arithmetic. He says the processes should be taught as memory or habit and any attempt to teach them on a rational basis is merely to stir up unnecessary trouble.

It is interesting to note the change in thinking of writers over a period of years on the subject. Morton[2] changed his views completely in a ten year period.

In 1927 he wrote:

> A large amount of practice should be provided at the time of first learning so that the bond may be fixed fairly well and then less and less amounts of practice should be provided at longer and longer intervals.

---

[1]Suzzalo, H., The Teaching of Primary Arithmetic, New York:Houghton, Miflin Company, Inc., 1911, pp. 60-64.

[2]Morton, R. L., op. cit., 1927, p. 39.

In 1937[1] he wrote:

> The bond psychology which was conspicuous in the earlier book has given away largely to psychology which emphasizes relationships and which recognizes that new elements may be discovered by the pupils.

Reed[2] writing at this time shows the same complete change in his thinking.

In 1927[3] he wrote:

"Arithmetic lends itself readily to training by repetition."

In a sequel written in 1938 he wrote:

"If the observation of relationships increases the ease of teaching, then we should expect that a method of teaching which makes relations easily perceptible would have an advantage in learning."

---

[1]Morton, R. L., Teaching Arithmetic in the Elementary School, New York:Silver Burdett Company, 1937, p. iv.

[2]Reed, H. B., op. cit., p. 122.

[3]Reed, H. B., Psychology of the Elementary School, New York:Silver Burdett Company, 1937, p. iv.

Not all students in the field of arithmetic have joined in the parade of support from the drill theory to the meaning theory. One of the early and ardent champions of an atomistic and mechanistic approach to the teaching of arithmetic holds the same point of view in his latest writings. Wilson[1] interprets the present trend in this manner.

> The present tendency is strongly toward the restoration of the original purpose of arithmetic, the organization of the subject around its use as a simple tool in business. If this aim is finally established, it will mean that we shall need to study arithmetic as drill.

Wilson's constant reference to "drill", the "drill load" and "100% mastery" are indicative of the point of view held by him and others in the early years of this century.

---

[1]Wilson, Guy M., Teaching the New Arithmetic, New York:McGraw Hill Book Company, 1939, p. 7.

The meaning of "meaning in arithmetic". Many educators tell us that arithmetic should be taught meaningfully. When we ask what is meant by teaching for "meaning" in arithmetic, we find that there are two major connotations of the word, social meaning and mathematical meaning. To clarify this situation Buckingham[1] uses the two words "significance" and "meaning" to refer to social meaning and mathematical meaning respectively.

Brownell[2] uses the terms similarly when he says that arithmetic skills do not become meaningful by using them in social situations. The learner develops only a significance of the usefulness of number in life. Meaning (mathematical) can be found only in the structure of the number system, through a study of number relationships and the rationale of computational processes.

---

[1]Buckingham, B. R., op. cit., p. 27.

[2]Brownell, W. A., "When Is Arithmetic Meaningful?", Journal of Educational Research, 38:481-98, March, 1945.

Since the appearance of these two papers there has been general acceptance of this distinction in terminology by most writers.

McConnell[1] says:

> There has been a tendency to assume that learning arithmetic in social situations and for social purposes makes it meaningful arithmetic. A moment's reflection will lead to the realization that specific training in the employment of an arithmetic procedure in a social situation may make no contribution whatever to the understanding of that process as such. The fact that oranges are attahhed to the formal repetition of 3 plus 4, for example, may have little to do with the child's insight into the number relations behind the verbalization. Fundamentally, to learn arithmetic meaningfully it is necessary to understand it systematically.

Throughout this study the words "meaning" and "meaningful" will be used to connote the mathematical meaning.

<u>The meaning theory defined</u>. Brownell[2] presents a concise summary of the presently evolving meaning theory.

---

[1]McConnell, T. R., <u>op. cit.</u>, p. 281.

[2]Brownell, W. A., National Council of Teachers of Mathematics, The Teaching of Arithmetic, 10th Yearbook, Chapter I, "Psychological Considerations in the Learning and Teaching of Arithmetic".

> . . . within the meaning theory there is absolutely no place for the view of arithmetic as a heterogeneous mass of unrelated elements to be trained through repetition. The meaning theory conceives of arithmetic as a closely knit system of understandable ideas, principles and processes. According to this theory, the test of learning is not mere mechanical facility in "figuring". The true test is an intellignet grasp upon number relations, and the ability to deal with arithmetic situations with proper comprehension of their mathematical as well as their practical significance.

Although Brownell designates this theory as the "meaning" theory, it should be noted that some writers prefer to use "generalization" theory, "organizational" theory, and "number relationship" theory. Throughout this study, these terms are used synonymously.

Nature and support of the meaning theory. As has been mentioned before the meaning theory is the original theory in the teaching and study of arithmetic, and it was basic to the study and writings of the Greek and Roman arithmeticians. This theory held throughout the middle ages and into the Renaissance. With the rise of commercialism came the need for large numbers of people skilled in computing, people who could get the correct answers "quickly and accurately".

The advent of mass education in America encouraged the production of workbooks, drill pads, practice sheets, etc., all of which encouraged the use of drill as the

major method of teaching. Arithmetic as a science of numbers, arithmetic as a system of related ideas and arithmetic as a body of information containing principles, understandings, and relationships assumed a minor role in the teaching of the subject. However, students of the subject continued to argue for a return to a more meaningful type of teaching.

One of the first books devoted entirely to the support of arithmetic as a system of related ideas is that of McLellan and Dewey[1] published in 1895. The preface presented their views.

> It is customary now to divide studies into 'form' studies and 'content' studies and to depreciate arithmetic on the ground that it is merely formal. But how are we to separate form and content and regard one as good in itself and the other as, at best, a necessary evil? An education which neglects the formal relationships constituting the framework of the subject matter is inert and supine" . .

---

[1]McLellan, J. A. and Dewey, J., op. cit., p. xii.

Barber[1] says:

". . . the understanding is the thing and not the quick and transitory ability to perform the operation . . ."

One of the most vigorous opponents of the atomistic approach and at the same time one of the strongest advocates of the meaning theory was Charles Judd[2]. He maintains that if teaching does not present the number system as a coherent and orderly system of thinking the learner is being deprived of one of the most important opportunities for scientific generalization. Arithmetic as a series of unrelated skills and abilities does not capitalize on the principle of generalization as an aid to memory and transfer.

Brownell[3] presents four arguments in support of the meaning theory.

1. Through the experiences of many teachers we know that meaningful arithmetic "works".

2. Since the drill arithmetic of the past years has failed, to develop arithmetical competence it is quite doubtful that a greater stress on the same method would help the situation.

---

[1]Barber, H. C., Teaching Junior High School Mathematics, New York:Houghton, Miflin Company, 1924.

[2]Judd, C. H., "Psychological Analysis of the Fundamentals of Arithmetic", Monograph #32, Department of Education, University of Chicago, 1927.

[3]Brownell, W. A., "The Place of Meaning in the Teaching of Arithmetic", The Elementary School Journal, 47:256-65, January, 1947.

3. Psychological research has presented findings without exception in favor of meaningful learning.

4. The meaning theory has been widely accepted in fields other than arithmetic so it is reasonable to assume that it would apply equally as well in arithmetic.

Summary statement. The previous pages have directed attention to the two major theories in the teaching of arithmetic. The drill theory is based upon connectionist psychology; the meaning theory is based on a psychology of generalization.

Proponents on both sides offer strong arguments in support of their particular theses but we should examine the research in the field before any conclusions are drawn.

Summary of research studies related to learning theories in arithmetic. Brownell[1] conducted one of the first studies. His was a detailed experimental study on the development of children's number ideas in the primary grades. Brownell used both group tests and the individual case study method as means for gathering data.

His conclusions were:

The reasonable course of action to adopt in teaching arithmetic would seem to be that which makes the largest possible use of children's capacity for generalization.

---

[1]Brownell, W. A., "The Development of Children's Number Ideas in the Primary Grades", Supplementary Educational Monograph #35, University of Chicago Dept. of Education, 1928, p. 215.

This scheme of instruction represents the opposite of specific teaching.

In a study made by Judd[1] at about this same time, the conclusions stress the value of generalization in the teaching and learning of arithmetic.

> The procedure of guiding the child to the complete understanding of number will be successful only when there is an intelligent analysis of the number system on the one hand and on the other an equally intelligent consideration of the child's modes of thinking and possibilities of development in the mastery of abstractions.

Woody[2] reports on a study which involved the development of arithmetic ability through the use of four different methods. In the first method, examples were taught as specific number situations, and no effort was made to accomplish more than to get the correct answer. In the second method the examples were taught with an effort to aid the child to develop a procedure useful for attacking similar problems. In the third method, an effort was made to generalize the procedure with the hope that the child could use it for solving other problems. In the

---

[1]Judd, C. H., "Psychological Analysis of the Fundamentals of Arithmetic", Supplementary Educational Monograph #32, University of Chicago, Dept. of Education, 1927.

[2]Woody, C., "Some Investigations Resulting from the Testing Program in Arithmetic", 17th Annual Conference on Educational Measurement. Bulletin of School of Education, Indiana University, 1930.

fourth method, the emphasis was placed on both generalization and rationalization. Pupils in fifty second grades were used as subjects. The practice material consisted of single columns in addition, three or four digits high, and some single column subtraction examples.

The transfer was measured by the success of the pupils in doing similar examples having two place and three place numbers. It was found that the method emphasizing generalization produced the greatest transfer. The method emphasizing both generalization and rationalization ranked second. Rationalization alone ranked third and specific instruction fourth.

The conclusions Woody draws are:

"This experiment indicates that the method of teaching influences the amount of transfer and that problems should be taught so that the pupils may not only get the answers but also learn the method."

In 1934 McConnell[1] conducted a controlled experiment in the learning of the 100 addition facts and the 100 subtraction facts. He used pupils in the second grade in the public schools of Toledo, Ohio.

---

[1]McConnell, T. R., "Discovery vs Authoritative Identification in the Learning of Children", Studies in the Psychology of Learning II. Educational Psychology Series #2, University of Iowa Studies in Education. Vol. IX, 1934.

Under "method A" pupils were taught the facts through repetition. No attempt was made to show relationships between the facts. Under "method B" pupils were taught the number facts through classroom practices which allowed the pupils to discover the facts for themselves. They were given help in seeing the relatedness of facts. The experiment extended over a seven month period. There was a total of 863 children in the collation of data at the end of the experiment. At the beginning of the experiment, pupils were equated on the basis of arithmetical scores and intelligence scores.

In interpreting his findings, McConnell says:

> If the teacher is interested in immediate and automatic response to the number fadts, the method of sheer repetition apparently can be counted upon to produce such a result with reasonable forthrightness. The meaningful procedures of method B apparently contribute little to the attainment of such an outcome. On the other hand, if the teacher desires to give the pupil whatever satisfaction may accrue to him through a knowledge of the meaning and truth of the number facts --- and to develop a deliberate and meditative attack on them, then method B commends itself. Although in this experiment method B led to a slight sacrifice of immediacy of response, there is also some evidence that it resulted in better ability to transfer learning and to manipulate the number facts in mature ways.

Thiele[1] conducted an investigation on the effect of generalization on the learning of addition facts and the effect of the drill method on the learning of the same facts. The duration of the teaching was fifteen weeks.

Thiele contrasted the two methods.

Generalization Method	Drill Method
1. Introduction of related addition combinations through social settings which involved the use of concrete material.	1. Same
2. Stage set in such a way that the perception of a useful generalization was required.	2. Introduction ended with the verification of the combinations.
3. Further application of the number combinations was made in drawing, dramatization and word problem exercises in which reference was made to generalization when necessary.	3. Application of the same kind of references was made to a combination chart whenever an answer was not recalled.
4. The perception of a useful generalization was checked and fixed through two digit combinations.	4. Repetitive drill on the combinations.
5. The number combinations were worked out on the number scale.	5. Repetitive drill on the combinations.
6. Review and practice exercises were conducted in which reference was made to generalization whenever necessary.	6. Review and practice exercises were directed in which memorization of the combinations was sought by the processes of repetitive drill.

---

[1]Thiele, C. L., "The Contribution of Generalization to the Learning of the Addition Facts", Teachers College Contribution to Education #763, 1938.

Thiele arrives at these conclusions:

1. For the study as a whole the differences are decidedly in favor of the generalization method. The mean gain made between the time of the initial and final tests by the generalization method pupils as a group is greater than the mean gain attained by the drill method pupils by an amount over eight times the standard deviation of the differences.

2. When the gains of smaller groups are compared on the basis of intelligence rating, the reliabilities of the differences are not so great but in every instance they approach 3.

3. When intelligence ratings are disregarded and comparisons are made on the basis of initial scores, the superiority of the generalization method is a significant one.

4. In the comparison of the per cent of possible gain, the generalization method results maintain their superiority.

5. Not only did the generalization method pupils make greater gains, but they also achieved higher averages when the relative difficulty of each combination was computed.

SUMMARY OF CHAPTER II

The research studies reported in this chapter indicate the superiority of instruction which aims at the development of meanings or generalizations. It would appear, then, that methods of instruction in use in schools should make for the development of mathematical understanding.

In Chapter III the structure of a test to determine within the limitations of the study, the degree of mastery

of specific basic mathematical understandings by junior high school mathematics pupils will be described.

CHAPTER III

PROCEDURE OF THE STUDY

Analysis of existing tests in mathematics.

Altogether too often growth in mathematical ability is thought to be synonymous with growth in computational skill. This evidence is seen in the form of workbooks, drill pads, increased amounts of drill material in textbooks and the testing materials. Teachers believe that the bulk of the mathematics program in the elementary grades should consist of the teaching of skills, the administering of a drill program and the measurement of facility in the skills. They believe a good remedial program consists of an elaborate testing program, the purpose of which is to determine the specific skills causing difficulty, and an equally elaborate program of providing more drill work in the skills to correct the difficulties. Most teachers believe that they have done a good job when they have succeeded in bringing the achievement of the average pupil in the class up to the grade norm on a standard achievement test. That this feeling should exist is entirely normal as there are few tests available to test anything other than computational skill or verbal problems in mathematics.

All tests that could be found which attempted to measure understandings were examined. As will be seen, there are few such tests available.

The Analytical Scales of Attainment Test[1] contains a section entitled "Quantitative Relationships". This is a new departure in the direction of understanding of numerical relationships. Morton[2] discusses this section in the 1941 Mental Measurements Yearbook.

". . . one may question whether an item on the composition of the dime which occurs in the test of Quantitative Relationships really measures the pupils' grasp of quantitative relationship at all."

The test seems to be a measure of arithmetical information. Sample questions are: "We should loan money only to people whom we . . . trust." "The figure of a buffalo is on the . . .nickel."

---

[1]Educational Test: Analytical Scales of Attainment, Kellogg, M., Brueckner, L. J. and Van Wagenen, M. J., 1933.

[2]Mental Measurements Yearbook, Buros, O. K., Highland Park, New Jersey, 1941.

The Faust Schorling Test of Functional Thinking in Mathematics[1] represents an attempt to measure understandings of mathematical relationships at upper grade levels. The authors state that it is for use on the high school and college levels. As it was not constructed for the purpose of measuring basic mathematical understandings, it does not contain items suitable for use in this study. The test contains 80 items of which 17 are True - False and 63 completion.

Butler constructed a test to measure the Mastery of Certain Mathematical Concepts by Pupils at the Junior High School Level[2]. This instrument was a well constructed work but measured the geometric phase of mathematics and contained no suitable items for this study.

The Iowa Every Pupil Test[3] contains one short section of multiple choice items. A sample question from this test is: "To change an improper fraction to a mixed number (1) divide the numerator by the denominator (2) divide the denominator by the numerator (3) divide both terms by 2."

---

[1]Faust, Schorling R., Test of Functional Thinking in Mathematics, World Book Company, New York, 1944.

[2]Butler, C. H., op. cit.

[3]Iowa Every Pupil Test: Iowa University, 1947.

Nolan constructed a test of Arithmetical Concept Thinking[1] for use at the junior high school level. This test was of the multiple choice variety and contained items in the same areas as are being considered in this study. Computation was necessary so none of the items were found to be useful although much helpful information was garnered from the reading of this thesis.

Of the few instruments that do contain items for measuring basic mathematical understandings that were studied, no single test contains enough items sufficiently specific or non-computational in form for use in this study.

Need for a new instrument. In Chapter II there was presented in part a summary of research in the field of learning of arithmetic carried out during the past few years, all of which led to the general conclusion that the development of basic mathematical meanings and understandings is an essential objective in the teaching of arithmetic. Accepting this as an aim carries with it the duty of measuring our success in the teaching of this aim,

---

[1]Nolan, J. A., The Construction of a Test of Arithmetical Concept Thinking, Master Thesis, Boston University, 1948.

and that in turn necessitates the use of new devices, instruments and methods. It is to a great extent due to the recency of this research and the acceptance of this objective that there are as yet no instruments available for measuring understandings.

Concerning the scarcity of tests designed to measure growth in mathematical understandings Butler[1] says:

". . . it is certain that in the domain of published tests the specific testing for mastery of mathematical concepts (understandings) has received scant attention."

Brownell[2] has this to say on the same point:

> Exceedingly little has been done either informally or systematically to find practicable and valid procedures for evaluating the outcomes under the heading above. (mathematical understandings) There are, for example, no standard tests available, except (a) two sections of the Analytical Scales of Attainment and (b) a shorter section in the Every Pupil Test. But these sections do not evaluate learning with respect to all the outcomes listed here under mathematical understandings. They do not evaluate fully or for all different purposes with respect to any one outcome listed.

---

[1]Butler, C. H., op. cit.

[2]Brownell, W. A., 16th Yearbook National Council of Teachers of Mathematics, Arithmetic in General Education, 1940, p. 247.

Chapter IX of the 15th Yearbook National Council of Teachers of Mathematics[1] contains several statements on this problem:

> The well trained mathematics teacher should also be interested in discovering the extent to which the pupil understands the significance of what he is doing. The measurement of this understanding calls for --- statements that explain why one proceeds as he does --- Questions of this type are sadly lacking in most tests and their absence is the basis for asserting that present day tests measure only restricted types of objectives even within the field of mathematics.(p.173)

> Testing in mathematics must find ways of measuring achievement of many objectives in addition to those dealing primarily with the recall of information and operational skills.(p.174)

> The development of means of evaluating achievement of important objectives not now measured is the first essential of the emerging program.(p.181)

Characteristics of the new instrument. The chief characteristic of the new instrument is the elimination entirely or the minimization of the effect of rote computational facility as a determiner of success. A test of mathematical understanding that involved computation habits learned largely through repetition would be quite invalid, since it would be difficult to determine the degree to which the testee's responses were the result of

---

[1] 15th Yearbook National Council of Teachers of Mathematics. The Place of Mathematics in Secondary Education.

understandings or the result of rote memory.

This statement does not carry the implication that tests of computational facility do not require understanding for success, or that pupils who do not understand will be as successful as those pupils who do understand. On the contrary, it is quite probable that the pupils with the greatest degree of understanding will be the most successful on the test of computational facility. However, it is true that the test which eliminates or minimizes computational facility will also eliminate one factor contributing to the nonvalidity of the instrument as a measure of understanding of number relationships. For this the new instrument is constructed in such a manner as to eliminate the necessity for direct computation.

Another characteristic of the instrument is objectivity. To attain this characteristic the test items were built in the form of multiple choice items. This type of item was chosen on the basis of such statements as that of Hawkes, Lindquist and Mann:[1]

---

[1]Hawkes, H. E., Lindquist, E. F., Mann, C. R., Construction and Use of Achievement Examinations, Boston:Houghton, Mifflin Company, 1926, p. 138.

"The multiple choice type is perhaps the most invaluable and the most generally applicable of all types of test exercises."

Greene, Jorgensen and Gerberich[1] make the same point:

"The multiple choice and its numerous variants perhaps represents the most widely applicable type of objective test items."

Construction of the Test of Basic Mathematical Understandings. As was stated in Chapter I, the five areas to be covered in the instrument were those areas considered by the National Council of Teachers of Mathematics as core topics of the elementary curriculum.

---

[1]Greene, H. A., Jorgensen, A. N., Gerberich, J. R., Measurements and Evaluations in Secondary Schools, New York:Longmans, Green and Company, 1943, p. 177.

1. Basic understandings of Place Value.

2. Basic understandings of the four fundamental processes with whole numbers.

3. Basic understandings of fractions and the four fundamental processes with fractions.

4. Basic understandings of decimals and the four fundamental processes with decimals.

5. Basic understandings of computations.

After deciding on these five fields, specific meanings and understandings in each of the areas were examined and multiple choice test items were designed to measure them. Ten questions in each area were constructed and carefully examined to determine that they were properly placed in the correct area. These fifty questions were then submitted to six elementary teachers (Grades 5-6) for their comments as to the understanding being a part of the arithmetic instruction in the elementary grades, the understanding being sufficiently important to be called basic, and any items in which lack of clarity might cause an incorrect response. As a result of this criticism, no questions had to be rejected because the understanding was not a part of the arithmetic instruction in the elementary grades and none of the questions was discarded because the understanding was not considered important enough to be called a basic mathematical understanding. Several responses

were changed at the suggestion of these teachers which made for greater clarity.

Validity of the items. Following the construction of the test came the task of validating the items. It was impossible to check the present list of items against another list as there was no other list of basic mathematical understandings available. This method of validation was, of necessity, discarded.

A second and widely used method of validation is that of determining the correlation between the scores made on this test and the scores made by the same testees on a criterion test. No criterion test is available for this comparison.

With both of these methods of validation eliminated, a third means of validation was employed. This method, in reality, is the best method of validation of any instrument, namely the observation of the behavior of the testee, keeping in mind the question, does the test item distinguish between the pupil who understands and the pupil who does not understand?

With this question in mind two children at a time in grade seven were contacted. It was felt that working with two pupils would tend to bring about a better rapport between

the testees and the testor. The method used to observe the behavior of the pupils consisted of having the children read a test item, select the correct response and then tell why they selected that response. Some questions were discovered that the children could answer correctly without really possessing the understanding involved in the item. These questions were revised and then submitted to a new pair of testees. The questions were given to ten pairs of children before the writer was satisfied that the items were being answered correctly by reason of understanding and not be guessing.

During the procedure the testor looked for evidence of reading difficulty stemming from the wording of the items. Few of these appeared and only slight rewording was necessary.

Administering the test. With the Test of Basic Mathematical Understanding (Appendix B) in final form it was submitted to 319 pupils; one hundred five of grade seven, one hundred forty-two of grade eight and seventy-two of grade nine. The test was administered by the writer. All pupils of grades seven and eight study mathematics but only those pupils who are taking the college course in grade nine

receive mathematical instruction. No time limit was set for the completion of the test, but every pupil completed the test during a period of forty minutes.

CHAPTER IV

ANALYSIS OF DATA

The data collected as a result of the Test of Basic Mathematical Understanding shows that Grade seven mastered 22.1% of the specific mathematical understandings asked, Grade eight mastered 24.6% of the understandings and Grade nine mastered 30.9%.

These results show the lack of understanding on the part of junior high school pupils. These understandings it must be remembered were taught in Grades one through six with further development and maintenance in Grades seven, eight and nine.

There seems to be a slight increase in understanding at successive grade levels although this difference is really not too great when one considers the pupils tested. The increase from Grade seven to Grade eight is 2.5% and from Grade eight to Grade nine 6.3%, but one must recall that the pupils of Grade nine taking the test are all pupils electing mathematics in preparation for college. The pupils of grades seven and eight would be average groups but those of Grade nine would have to be classified as select.

The material collected will be presented in table form with Table I representing the number of responses correct and the percentage of correct responses for each item at the seventh grade level. Table II represents the number of responses correct and the percentage of correct responses for each item at the eighth grade level. Table III represents the number of responses correct and the percentage of correct responses for each item at the ninth grade level. Table IV represents the order of difficulty of the basic mathematical understandings in grade seven. Table V represents the order of difficulty of the basic mathematical understandings in grade eight. Table VI represents the order of difficulty of the basic mathematical understandings in grade nine. Table VII combines Tables I, II and III for the convenience of the reader. Table VIII combines Tables IV, V and VI for the convenience of the reader. Table IX represents the number of correct responses and the percentage of correct responses for each section of the Test of Basic Mathematical Understandings.

At the seventh grade level there were no questions receiving a 100% correct response by the testees. The nearest approach was 86% for question 11. There were no correct responses to 13 of the items. Out of 5,250

responses, 1.158 were answered correctly for a percentage of 22.1%. Breaking these correct responses down for the five sections into which the instrument was divided shows that there were 427 correct responses out of 1,050 or 40.6% for Section 1 (Basic Understandings of Place Value); for Section 2 (Basic Understandings of the Four Fundamental Processes with Whole Numbers) 376 correct responses out of 1,050 or 35.8%; for Section 3 (Basic Understandings of Fractions and the Four Fundamental Processes with Fractions) 191 correct responses out of 1,050 or 18.2%; for Section 4 (Basic Understandings of Decimals and the Four Fundamental Processes with Decimals) 83 correct responses our of 1,050 or 7.9%; for Section 5 (Basic Understandings of Computation) 81 correct responses out of 1,050 or 7.7%.

At the eighth grade level there were no questions receiving a 100% correct response. The nearest approach was 94% to question 11. There were no correct responses to 10 of the items. Out of 7,100 responses, 1,744 were answered correctly for a percentage of 24.6%. Breaking these correct responses down for the five sections of the test shows that Section 1 had 611 correct responses out of 1,420 or 43.0%; Section 2, 561 correct responses out of 1,420 or 39.5%; Section 3, 359 correct responses out of

1,420 or 25.3%; Section 4, 150 correct responses out of 1,420 or 10.6%; Section 5, 63 correct responses out of 1,420 or 4.4%.

At the ninth grade level there were no questions receiving a 100% correct response. The nearest approach was 96% for question 11. There were no correct responses to 4 of the items. Out of 3,600 responses, 1,113 were answered correctly for a percentage of 30.9%. Breaking these correct responses down for the five sections of the test shows that Section 1 had 332 correct responses out of 720 or 46.1%; Section 2, 368 correct responses out of 720 or 51.1%; Section 3, 190 correct responses out of 720 or 26.4%; Section 4, 191 correct responses out of 720 or 26.5%; Section 5, 32 correct responses out of 720 or 4.4%.

Section 1, Basic Understandings of Place Value, was the easiest for the pupils of grade seven and eight and ranked second in grade nine. Section 2, Basic Understandings of the Four Fundamental Processes with Whole Numbers, was the second easiest for grades seven and eight and was easiest for grade nine. Section 3, Basic Understandings of Fractions and the Four Fundamental Processes with Fractions, was third easiest for grades seven and eight and fourth easiest for grade nine. Section 4,

Basic Understandings of Decimals and the Four Fundamental Processes with Decimals was fourth easiest for grades seven and eight and third easiest for grade nine. Section 5, Basic Understandings of Computations was hardest for all grades.

In grade seven there was only a slight difference between the number of correct responses on Section 4 and 5. In grade nine there was only a slight difference between Section 3 and 4. It should be noted that grade seven did much better on Section 5 than did grades eight and nine if one can conceive of a 7.7% response being a good response; it would be more appropriate to say, did less poorly.

TABLE I

NUMBER OF CORRECT RESPONSES
AND PERCENTAGE OF CORRECT
RESPONSES IN GRADE SEVEN

Item	No. of Correct Responses	Percentage of Correct Responses
1	89	85
2	75	71
3	73	69
4	45	43
5	38	36
6	56	53
7	24	23
8	--	--
9	27	26
10	--	--
11	90	86
12	56	53
13	55	53
14	11	10
15	3	3
16	16	15
17	6	6
18	--	--
19	72	68
20	67	64
21	59	56
22	20	19
23	59	56
24	16	15
25	11	10
26	--	--
27	7	7
28	--	--
29	19	18
30	--	--

TABLE I (continued)

NUMBER OF CORRECT RESPONSES
AND PERCENTAGE OF CORRECT
RESPONSES IN GRADE SEVEN

Item	No. of Correct Responses	Percentage of Correct Responses
31	18	17
32	17	16
33	--	--
34	19	18
35	1	1
36	--	--
37	6	6
38	1	1
39	18	17
40	3	3
41	37	35
42	16	15
43	2	2
44	--	--
45	--	--
46	--	--
47	--	--
48	--	--
49	12	11
50	14	13

TABLE II

NUMBER OF CORRECT RESPONSES
AND PERCENTAGE OF CORRECT
RESPONSES IN GRADE EIGHT

Item	No. of Correct Responses	Percentage of Correct Responses
1	129	91
2	114	80
3	82	58
4	70	49
5	61	43
6	85	60
7	23	16
8	--	--
9	47	33
10	--	--
11	133	94
12	87	61
13	97	68
14	13	9
15	7	5
16	17	12
17	11	8
18	--	--
19	111	78
20	85	60
21	129	91
22	51	36
23	90	63
24	22	16
25	29	21
26	--	--
27	9	6
28	--	--
29	29	21
30	--	--

TABLE II (continued)

NUMBER OF CORRECT RESPONSES
AND PERCENTAGE OF CORRECT
RESPONSES IN GRADE EIGHT

Item	No. of Correct Responses	Percentage of Correct Responses
31	21	15
32	36	25
33	--	--
34	30	21
35	4	3
36	--	--
37	14	10
38	14	10
39	16	11
40	15	11
41	20	14
42	3	2
43	--	--
44	1	1
45	8	6
46	2	1
47	--	--
48	--	--
49	10	7
50	19	14

TABLE III

NUMBER OF CORRECT RESPONSES
AND PERCENTAGE OF CORRECT
RESPONSES IN GRADE NINE

Item	No. of Correct Responses	Percentage of Correct Responses
1	67	93
2	54	75
3	47	65
4	38	52
5	38	53
6	37	52
7	25	35
8	1	1
9	22	30
10	3	6
11	69	96
12	61	85
13	49	69
14	15	20
15	12	16
16	10	14
17	24	33
18	--	--
19	68	95
20	60	83
21	43	60
22	20	28
23	55	76
24	19	26
25	8	11
26	1	1
27	15	20
28	--	--
29	18	25
30	11	15

TABLE III (continued)

NUMBER OF CORRECT RESPONSES
AND PERCENTAGE OF CORRECT
RESPONSES IN GRADE NINE

Item	No. of Correct Responses	Percentage of Correct Responses
31	50	69
32	40	56
33	3	5
34	9	13
35	17	23
36	1	1
37	7	9
38	14	19
39	34	47
40	16	22
41	12	17
42	4	8
43	--	--
44	1	1
45	1	1
46	7	10
47	--	--
48	2	3
49	2	3
50	3	5

TABLE IV

ORDER OF DIFFICULTY OF BASIC
MATHEMATICAL UNDERSTANDINGS
OF GRADE SEVEN

	Item Number in Test	Number of Correct Responses	Percentage of Correct Responses
1	11	90	86
2	1	89	85
3	2	75	71
4	3	73	69
5	19	72	68
6	20	67	64
7	21	59	56
8	23	59	56
9	12	56	53
10	6	56	53
11	13	55	53
12	4	45	43
13	5	38	36
14	41	37	35
15	9	27	26
16	7	24	23
17	22	20	19
18	34	19	18
19	29	19	18
20	39	18	17
21	31	18	17
22	32	17	16
23	24	16	15
24	16	16	15
25	42	16	15
26	50	14	13
27	49	12	11
28	14	11	10
29	25	11	10
30	27	7	7

TABLE IV (continued)

ORDER OF DIFFICULTY OF BASIC
MATHEMATICAL UNDERSTANDINGS
OF GRADE SEVEN

Item Number in Test		Number of Correct Responses	Percentage of Correct Responses
31	17	6	6
32	37	6	6
33	40	3	3
34	15	3	3
35	43	2	2
36	35	1	1
37	38	1	1
38	30		
39	46		
40	45		
41	10		
42	33		
43	48		
44	44		
45	8		
46	26		
47	36		
48	18		
49	28		
50	47		

TABLE V

ORDER OF DIFFICULTY OF BASIC
MATHEMATICAL UNDERSTANDINGS
OF GRADE EIGHT

	Item Number in Test	Number of Correct Responses	Percentage of Correct Responses
1	11	133	94
2	1	129	91
3	21	129	91
4	2	114	80
5	19	111	78
6	13	97	68
7	23	90	63
8	12	87	61
9	20	85	60
10	6	85	60
11	3	82	58
12	4	70	49
13	5	61	43
14	22	51	36
15	9	47	33
16	32	36	25
17	34	30	21
18	25	29	21
19	29	29	21
20	7	23	16
21	24	22	16
22	31	21	15
23	41	20	14
24	50	19	14
25	16	17	12
26	39	16	11
27	40	15	11
28	38	14	10
29	37	14	10
30	14	13	9

TABLE V (continued)

ORDER OF DIFFICULTY OF BASIC
MATHEMATICAL UNDERSTANDINGS
OF GRADE EIGHT

Item Number in Test		Number of Correct Responses	Percentage of Correct Responses
31	17	11	8
32	49	10	7
33	27	9	6
34	45	8	6
35	15	7	5
36	35	4	3
37	42	3	2
38	46	2	1
39	26	1	1
40	44	1	1
41	30		
42	10		
43	33		
44	48		
45	43		
46	8		
47	36		
48	18		
49	28		
50	47		

TABLE VI

ORDER OF DIFFICULTY OF BASIC MATHEMATICAL UNDERSTANDINGS OF GRADE NINE

	Item Number in Test	Number of Correct Responses	Percentage of Correct Responses
1	11	69	96
2	19	68	95
3	1	67	93
4	12	61	85
5	20	60	83
6	23	55	76
7	2	54	75
8	31	50	69
9	13	49	69
10	3	47	65
11	21	43	60
12	32	40	56
13	5	38	53
14	4	38	52
15	6	37	52
16	39	34	47
17	7	25	35
18	17	24	33
19	9	22	30
20	22	20	28
21	24	19	26
22	29	18	25
23	35	17	23
24	40	16	22
25	14	15	20
26	27	15	20
27	38	14	19
28	41	12	17
29	15	12	16
30	30	11	15

TABLE VI (continued)

ORDER OF DIFFICULTY OF BASIC
MATHEMATICAL UNDERSTANDINGS
OF GRADE NINE

	Item Number in Test	Number of Correct Responses	Percentage of Correct Responses
31	16	10	14
32	34	9	13
33	25	8	11
34	46	7	10
35	37	6	9
36	42	6	8
37	10	4	6
38	50	3	5
39	33	3	5
40	48	2	3
41	49	2	3
42	36	1	1
43	45	1	1
44	8	1	1
45	26	1	1
46	44	1	1
47	43		
48	18		
49	28		
50	47		

TABLE VII

NUMBER OF CORRECT RESPONSES AND PERCENTAGE OF CORRECT RESPONSES FOR EACH ITEM IN GRADES SEVEN, EIGHT AND NINE

Item	Number of Correct Responses Grades			Percentage of Correct Responses-Grades		
	7	8	9	7	8	9
1	89	129	67	85	91	93
2	75	114	54	71	80	75
3	73	82	47	69	58	65
4	45	70	38	43	49	52
5	38	61	38	36	43	53
6	56	85	37	53	60	52
7	24	23	25	23	16	35
8	--	--	1	--	--	1
9	27	47	22	26	33	30
10	--	--	3	--	--	6
11	90	133	69	86	94	96
12	56	87	61	53	61	85
13	55	97	49	53	68	69
14	11	13	15	10	9	20
15	3	7	12	3	5	16
16	16	17	10	15	12	14
17	6	11	24	6	8	33
18	--	--	--	--	--	--
19	72	111	68	68	78	95
20	67	85	60	64	60	83
21	59	129	43	56	91	60
22	20	51	20	19	36	28
23	59	90	55	56	63	76
24	16	22	19	15	16	26
25	11	29	8	10	21	11
26	--	--	1	--	--	1
27	7	9	15	7	6	20
28	--	--	--	--	--	--
29	19	29	18	18	21	25
30	---	--	11	--	--	15

TABLE VII (continued)

NUMBER OF CORRECT RESPONSES AND PERCENTAGE OF CORRECT RESPONSES FOR EACH ITEM IN GRADES SEVEN, EIGHT AND NINE

Item	Number of Correct Responses Grades			Percentage of Correct Responses-Grades		
	7	8	9	7	8	9
31	18	21	50	17	15	69
32	17	36	40	16	25	56
33	--	--	3	--	--	5
34	19	30	9	18	21	13
35	1	4	17	1	3	23
36	--	--	1	--	--	1
37	6	14	7	6	10	9
38	1	14	14	1	10	19
39	18	16	34	17	11	47
40	3	15	16	3	11	22
41	37	20	12	35	14	17
42	16	3	4	15	2	8
43	2	--	--	2	--	--
44	--	1	1	--	1	1
45	--	8	1	--	6	1
46	--	2	7	--	1	10
47	--	--	--	--	--	--
48	--	--	2	--	--	3
49	12	10	2	11	7	3
50	14	19	3	13	14	5

TABLE VIII

ORDER OF DIFFICULTY OF
BASIC MATHEMATICAL
UNDERSTANDINGS

Item	No. in Test Grade 7	8	9	No. Correct Responses Grade 7	8	9	% Correct Responses Grade 7	8	9
1	11	11	11	90	133	69	86	94	96
2	1	1	19	89	129	68	85	91	95
3	2	21	1	75	129	67	71	91	93
4	3	2	12	73	114	61	69	80	85
5	19	19	20	72	111	60	68	78	83
6	20	13	23	67	97	55	64	68	76
7	21	23	2	59	90	54	56	63	75
8	23	12	31	59	87	50	56	61	69
9	12	20	13	56	85	49	53	60	69
10	6	6	3	56	85	47	53	60	65
11	13	3	21	55	82	43	53	58	60
12	4	4	32	45	70	40	43	49	56
13	5	5	5	38	61	38	36	43	53
14	41	22	4	37	51	38	35	36	52
15	9	9	6	27	47	37	26	33	52
16	7	32	39	24	36	34	23	25	47
17	22	34	7	20	30	25	19	21	35
18	34	25	17	19	29	24	18	21	33
19	29	29	9	19	29	22	18	21	30
20	39	7	22	18	23	20	17	16	28
21	31	24	24	18	22	19	17	16	26
22	32	31	29	17	21	18	16	15	25
23	24	41	35	16	20	17	15	14	23
24	16	50	40	16	19	16	15	14	22
25	42	16	14	16	17	15	15	12	20
26	50	39	27	14	16	15	13	11	20
27	49	40	38	12	15	14	11	11	19
28	14	38	41	11	14	12	10	10	17
29	25	37	15	11	14	12	10	10	16
30	27	14	30	7	13	11	7	9	15

TABLE VIII (continued)

ORDER OF DIFFICULTY OF
BASIC MATHEMATICAL
UNDERSTANDINGS

Item	No. in Test Grade			No. Correct Responses Grade			% Correct Responses Grade		
	7	8	9	7	8	9	7	8	9
31	17	17	16	6	11	10	6	8	14
32	37	49	34	6	10	9	6	7	13
33	40	27	25	3	9	8	3	6	11
34	15	45	46	3	8	7	3	6	10
35	43	15	37	2	7	6	2	5	9
36	35	35	42	1	4	6	1	3	8
37	38	42	10	1	3	4	1	2	6
38	30	46	50		2	3		1	5
39	46	26	33		1	3		1	5
40	45	44	48		1	2		1	3
41	10	30	49			2			3
42	33	10	36			1			1
43	48	33	45			1			1
44	44	48	8			1			1
45	8	43	26			1			1
46	26	8	44			1			1
47	36	36	43						
48	18	18	18						
49	28	28	28						
50	47	47	47						

TABLE IX

NUMBER OF CORRECT RESPONSES AND PERCENTAGE OF CORRECT RESPONSES FOR EACH SECTION OF TEST OF BASIC MATHEMATICAL UNDERSTANDING

	Number of Correct Responses			Percentage of Correct Responses		
	Grade			Grade		
	7	8	9	7	8	9
Section 1	427	611	332	40.6	43.0	46.1
Section 2	376	561	368	35.8	39.5	51.1
Section 3	191	359	190	18.2	25.3	26.4
Section 4	83	150	191	7.9	10.6	26.5
Section 5	81	63	32	7.7	4.4	4.4
	1,158	1744	1113	22.1	24.6	30.9

CHAPTER V

SUMMARY AND CONCLUSIONS

Summary of the Study

The purpose of this study was to determine to what extent specific basic mathematical understandings taught in grades one through six are mastered by students in grades seven through nine in courses of mathematics in the public schools of Milton, Massachusetts.

Teachers are in general agreement with the statement that arithmetic should be taught meaningfully. However, there is little understanding on the part of most teachers of the word "meaningful" as applied to the teaching of arithmetic. Most teachers consider it to be synonymous with "socially useful".

This general lack of knowledge of the meaning of the term is widely reflected in the teaching of arithmetic. By far the most prevalent activity in use in the arithmetic lesson is the repetitive or drill activity, the purpose of which is to bring about a degree of facility or skill. Such an activity does not and cannot contribute in any appreciable amount to the development of meanings or understandings.

In the last few years the word "meaningful" has taken on a very definite connotation, a connotation that is quite the opposite of "socially useful". It implies a type of teaching in which an attempt is made to develop an understanding of the relationships existing between processes in the number system. It is this understanding and approach to the teaching of arithmetic which most teachers lack.

While the acceptance of teaching for meaning and understanding has been acknowledged, and schools have put these teachings into operation, there has not as yet developed a means of measuring the success of the teachings. It was due to the lack of such an instrument that the Test of Basic Mathematical Understanding was constructed.

Conclusions of the Study

An analysis of the 15,950 responses furnishes the data from which the following conclusions are drawn.

1. Grade seven mastered 22.1% of the understandings basic to computational processes taught in grades one through six.

2. Grade eight mastered 24.6% of the understandings basix to computational processes taught in grades one through six.

3. Grade nine mastered 30.9% of the understandings basic to computational processes taught in grades one through six.

4. There is an increase in achievement of basic mathematical understanding on increasing grade levels.

5. The understandings that are difficult in one grade are difficult in the other grades; the understandings that are easy in one grade are easy in the other grades.

6. The understandings of place value and of the fundamental processes with whole numbers were easiest for testees.

In spite of the findings of research studies pointing to the desirability of teaching for understanding, the above conclusions show that the persons tested have not acquired a satisfactory knowledge of the understandings involved in elementary arithmetic. The pupils tested must have computational skill in arithmetic as witnessed by their promotion from grade to grade above the elementary level but this skill does not bring about understanding.

The above conclusions have profound implications for principles of teaching and curriculum development in arithmetic. The method of teaching should emphasize the development

of the meanings and understandings inherent in the number system. This shift in emphasis would bring a concomitant decrease in emphasis on the mastery of isolated computational skills. The teaching should stress relatedness rather than atomization. Learning activites and experiences should be numerous and varied rather than few and striated.

BIBLIOGRAPHY

BIBLIOGRAPHY

Bailey, M. A., "The Thorndike Philosophy of Teaching the Processes and Principles of Arithmetic," Mathematics Teacher, March, 1923.

Barber, H. C., Teaching Junior High School Mathematics, New York: Houghton, Miflin Company, 1924.

Betz, William, "The Necessary Redirection of Mathematics; Including Its Relation to National Defense," Mathematics Teacher, 35:147-160, April, 1942.

Brownell, William, "When Is Arithmetic Meaningful?", Journal of Educational Research, 38:481-98, March, 1945.

Brownell, William, "Essential Mathematics for Minimum Army Needs," School Review,52:484-92. October, 1944.

Brownell, William, National Council of Teachers of Mathematics. The Teaching of Arithmetic, 10th Yearbook. Psychological Considerations in the Learning and Teaching of Arithmetic. New York: Bureau of Publications, Teachers College Columbia University, 1935.

Brownell, William, National Council of Teachers of Mathematics, 16th Yearbook. Arithmetic in General Education. New York: Bureau of Publications, Teachers

Brownell, William, "The Place of Meaning in the Teaching of Arithmetic," The Elementary School Journal,47:256-65, January, 1947.

Brownell, William, The Development of Children's Number Ideas in the Primary Grades, Supplementary Educational Monograph #35, University of Chicago, Department of Education, 1928.

Brueckner, L. J., Diagnostic and Remedial Teaching in Arithmetic, Philadelphia: J. C. Winston Company, 1930.

Buckingham, B. R., "Significance, Meaning and Insight, These Three," The Mathematics Teacher, 31:24-30, January, 1938.

Buckingham, B. R., "The Contribution of Arithmetic to a Liberal Education," The Mathematics Teacher, 35:51-8, February, 1942.

Buell, Irwin A., "Let Us Be Sensible About It," The Mathematics Teacher, 37:306-8, November, 1944.

Buros, O.,K., Mental Measurements Yearbook. Highland Park, New Jersey, 1941.

Buswell, Guy T., Science and Mathematics, 43;201-12, March, 1943.

Butler, C. H., "Mastery of Certain Mathematical Concepts by Pupils at the Junior High School Level," The Mathematics Teacher, 25:117-172, March, 1932.

Colburn, Warren, Intellectual Arithmetic Upon the Inductive Method. Concord: Sanborn and Company, 1840.

Dewey, John, How We Think. Boston: D. C. Heath and Company, 1933.

Dickey, J. W., "Arithmetic and Gestalt Psychology," Elementary School Journal, 39:46-53, September, 1938.

Faust, Schorling, Test of Functional Thinking in Mathematics. New York: World Book Company, 1944.

General Education in a Free Society. Report of the Harvard Committee. Cambridge: Harvard University Press, 1945.

Greene, H. A., Jorgensen, A. N., and Gerberich, J. R., Measurements and Evaluations in Secondary Schools. New York: Longmans, Green and Company, 1943.

Hawles, H. E., Lindquist, E. F., and Mann, C. R., Construction and Use of Achievement Examinations. Boston: Houghton Mifflin Company, 1936.

Hullfish, H. G., Aspects of Thorndike's Psychology in Their Relation to Educational Theories and Practices, Ohio State University Studies #1. Columbus, Ohio, 1926.

Judd, Charles H., Psychological Analysis of the Fundamentals of Arithmetic, Nonograph #32, Department of Education, University of Chicago, 1927.

Judd, Charles H., Education as Cultivation of the Higher Mental Processes. New York: MacMillan Company, 1936.

Kellog, M., Brueckner, L. J., and Van Wagenen, Analytical Scales of Attainment. Educational Test, 1933.

Mac Latchy, J. H., "Seeing and Understanding in Number," Elementary School Journal, 45:144-52, November, 1944.

McConnell, T. R., National Council of Teachers of Mathematics, 16th Yearbook. Arithmetic in General Education. Chapter XI, "Recent Trends in Learning Theory", pp. 268-289.

McConnell, T. R., Discovery vs Authoritative Identification in the Learning of Children. Studies in the Psychology of Learning. Educational Psychology Series #2, University of Iowa, Studies in Education, 1934.

McLellan, J. A. and Dewey, J., The Psychology of Number. New York: D. Appleton and Company, 1895.

Mossman, E. L., "Which Shall It Be: Mechanical Drill or Development in Understanding All the Whys?" The Mathematics Teacher, 38:103-7, March, 1945.

National Society for the Study of Education, The Measurement of Understanding, 45th Yearbook. Bloomington, Illinois: Public School Publishing Company, 1946.

National Council of Teachers of Mathematics, The Place of Mathematics in Secondary Education, 15th Yearbook. Bureau of Publications, Teachers College, Columbia University, 1940.

Nolan, J. A., The Construction of a Test of Arithmetical Concept Thinking. Master's Thesis, Boston University, 1948.

Reed, H. B., Psychology of Elementary School Subjects. Boston: Ginn and Company, 1927.

Reed, H. B., Psychology of the Elementary School. New York: Silver Burdett and Company, 1937.

Riess, A., "Meaning of the Meaningful Teaching of Arithmetic," Elementary School Journal, 45:23-32, September, 1944.

Suelz, Ben A., "The Measurement of Understandings and Judgments of Elementary School Mathematics," The Mathematics Teacher, May, 1947.

Suzzallo, H., The Teaching of Primary Arithmetic. New York: Houghton, Miflin Company, Inc., 1911.

Thiele, C. L., The Contribution of Generalization to the Learning of the Addition Facts. Teachers College Contribution to Education #763, 1938.

Thorndike, E. L., Psychology of Arithmetic. New York: MacMillan Company, 1932.

Wheeler, L. R., "A Comparative Study of Difficulty of the 100 Addition Combinations." Journal of Genetic Psychology, 54:205-312, June, 1939.

Wilson, Guy M., Mildred B. Stone, Charles O. Dalrymple/Teaching the New Arithmetic. New York: McGraw Hill Book Company, 1939.

Woody, Clifford, Some Investigations Resulting from the Testing Program in Arithmetic, 17th Annual Conference on Educational Measurement. Bulletin of School of Education. Indiana University, 1930.

APPENDIX A

LIST OF BASIC MATHEMATICAL UNDERSTANDINGS MEASURED BY TEST ITEMS

BASIC MATHEMATICAL UNDERSTANDINGS MEASURED BY TEST ITEMS

SECTION ONE

BASIC UNDERSTANDINGS OF PLACE VALUE

1. In a group of numbers with the same digits but different values, that number has the smallest value which has the smallest figure in the position of highest value, the next smallest figure in the position of next highest value, etc. with the largest figure in the position of least value.

2. In a group of numbers with the same digits but different values, that number has the largest value which has the largest figure in the position of highest value, the next largest figure in the position of next highest value, etc. with the smallest figure in the position of least value.

3. The third place to the left of the decimal point is the hundred's place.

4. A digit in the fourth place to the left of the decimal point represents a number 1,000 times as large as the same digit in the first place to the left of the decimal point.

5. Any number larger than 10 contains as many tens as there are represented by the digits in the places to the left of the units place.

6. The fourth place to the left of the decimal point is the thousands place.

7. Zero is used as a place holder when there is no frequency to record in any given position in a number.

8. A digit in the first place represents a number one-tenth as large as the same digit in the second place to the left of the decimal point.

9. The value of a digit in a number depends upon its position in the number and its numerical value.

10. A digit in the tens place that has half the absolute value of a digit in the hundreds place represents a number of one-twentieth its value.

SECTION TWO

BASIC UNDERSTANDINGS OF THE FOUR FUNDAMENTAL PROCESSES WITH WHOLE NUMBERS

11. Division is a quick method for determining the size of equal parts into which a number is divided.

12. When a whole number is multiplied by a whole number other than 1, the answer is larger than the number multiplied.

13. When a whole number is divided by a whole number other than 1, the answer has a numerical value smaller than the number divided.

14. In a subtraction example the sum of the subtrahend and remainder equals the minuend.

15. Adding two zeros to the right of a whole number has the effect of multiplying the number by 100.

16. Multiplying the multiplicand by 100 and dividing the multiplier by 10 has the effect of increasing the product 10 times.

17. Crossing off a zero from the right side of a whole number has the same effect as dividing the number by 10.

18. Multiplying the divisor by 100 and dividing the dividend by 10 has the effect of dividing the quotient by 1000.

19. Changing the order of addends in an addition example does not change the answer.

20. Reversing the positions of the multiplier and the multiplicand does not change the value of the product.

SECTION THREE

BASIC UNDERSTANDINGS OF THE FOUR FUNDAMENTAL PROCESSES WITH FRACTIONS

21. In a number of fractions having the same denominators, the fraction having the largest denominator has the smallest value.

22. The denominator of a fraction tells the number of equal parts into which the whole is divided.

23. In a number of fractions having the same numerator, the fraction having the largest denominator has the smallest value.

24. When a whole number is multiplied by a proper fraction, the answer has a numerical value smaller than the original number.

25. When a proper fraction is divided by a proper fraction, the answer has a numerical value larger than the fraction divided.

26. Multiplying the denominator of a fraction by 2 gives a fraction that is one half of the value of the original fraction.

27. When a whole number is divided by a proper fraction, the answer has a numerical value larger than the whole number.

28. When dividing a whole number by a fraction, the answer has a numerical value larger than the whole number because the divisor is less than 1.

29. Dividing the numerator and denominator of a fraction by the same number does not change the value of the fraction.

30. When a proper fraction is multiplied by a proper fraction, the answer has a numerical value smaller than the fraction multiplied.

SECTION FOUR

BASIC UNDERSTANDINGS OF THE FOUR FUNDAMENTAL PROCESSES WITH DECIMALS

31. When writing the mixed decimal "eighty and eight hundredths" the "eighty" is written as a whole number and "and" is represented by a decimal point, and the "eight hundredths" is written as a two place decimal, (.08) with a zero holding the first decimal place.

32. When reading the decimal .0309, the value is expressed as ten-thousandths.

33. The fraction 5/8 can be expressed as the decimal .625.

34. In a multiplication example involving the decimal 23.90 the zero may be left off without changing the value of the answer.

35. The example "ten divided by five-tenths" can be read "how many halves are there in ten".

36. In a series of decimals the largest decimal is the one that has the largest digit in the decimal place of greatest value.

37. In a multiplication example we may move the point one place to the right in the multiplicand and move the point one place to the left in the multiplier without changing the value of the answer.

38. In a multiplication example moving the point one place to the left in both numbers would make the answer one-hundredth as large as the answer in the original example.

39. In a division example moving the point one place to the right in the dividend would make the answer ten times as large as the answer to the original example.

40. When multiplying a whole number by a decimal the answer is smaller than the whole number because the decimal has a value less than one.

SECTION FIVE

BASIC UNDERSTANDINGS OF COMPUTATIONS

41. The process of finding a common denominator is a method for changing units of different size as represented by the denominators into units of the same size.

42. When dividing by a decimal, we first change the divisor to a whole number by multiplying by the appropriate power of ten, multiply the dividend by the same power. Dividing by a whole number is easier than dividing by a decimal since it is necessary to control only one decimal point.

43. In the example 2.1 x 21 the product is the sum of 1 x 2.1 and 20 x 2.1.

44. Dividing by a fraction is the same as multiplying by the reciprocal of the fraction.

45. When multiplying by a figure in the tens place, we move the second partial product one place to the left in order to leave a space for the zero.

46. Arranging addends in column helps one to add together only those numbers of the same decimal value.

47. When multiplying by 36, the second partial product represents a number five times as large as the first partial product.

48. In a division example if the first digit in the quotient is a 4 located in the tens place, the dividend will contain the divisor at least 40 times.

49. If it is necessary to borrow when subtracting a fraction from a mixed number, the "one" that is borrowed is changed into a fraction with the same denominator as the common denominator and the new fraction is added to the fraction in the minuend.

50. When a fraction is reduced to lowest terms, the value of the fraction does not change, but the terms of the fraction are smaller.

APPENDIX B

A TEST OF BASIC MATHEMATICAL UNDERSTANDINGS

A TEST OF BASIC MATHEMATICAL UNDERSTANDINGS

Directions:

This is a test to see how well you understand arithmetic. You do not have to do any written work to determine the answers.

Read each statement carefully and decide which of the answers is the correct answer. Write the letter for this answer on the line at the right of the example.

Sample Item.

Which of the following numbers has the largest value?

(a) 23 (b) 9 (c) 35 (d) 45 (e) 11 ___d___

45 is the correct answer so we write (d) on the line at the right.

Try each example, but don't spend too much time on any one example. If you can't find the answer, go to the next example.

Shaded pictures are read like this:

= 1/4

REMEMBER - DO NO WRITTEN WORK TO GET THE ANSWER

Basic Understandings of Place Value

1. If you rearrange the figures in the number 97,854, which of the following arrangements would give you the smallest number? _____
(a) 45,789 (b) 74,895 (c) 54,789 (d) 45,879 (e) 98,754

2. If you rearrange the figures in the number 65,327 which of the following arrangements would give you the largest number? _____
(a) 23,567 (b) 65,327 (c) 67,532 (d) 76,532 (e) 53,276

3. Which of the following numbers has a 3 in the hundred's place? _____
(a) 43,675 (b) 56,934 (c) 49,723 (d) 62,345 (e) 37,982

4. In the number 5,555 the 5 on the left represents a value how many times as large as the 5 on the right? _____
(a) 100 (b) same value (c) 10 (d) 500 (e) 1000

5. How many tens are there in the number 3740? _____
(a) 3.74 (b) 37 2/5 (c) 374 (d) 3740 (e) 37.4

6. If the figures in the number 42,567 were rearranged, which of the following would place the largest figure in the thousand's place? _____
(a) 67,254 (b) 76,542 (c) 24,567 (d) 45,672 (e) 54,762

7. Which of the following statements best tells why we write a zero in the number 5.076? _____
(a) Writing the zero helps in reading the number.
(b) The number would be wrong if we left out the zero.
(c) Writing the zero tells us not to read the hundred's figure.
(d) The number would read five hundred seventy-six if the zero is left out.
(e) Writing the zero as a place holder shows that there is no number to record in that place.

8. In the number 5688, the 8 on the right represents a value how many times as large as the 8 on the left? _____
(a) 1/2 (b) 1/10 (c) same value (d) 1/8 (e) 10

9. Which of the following methods is best used to determine the value of a figure in a number? _____
(a) Its position in the number.
(b) Its value when compared with other figures in the number.
(c) Its value in the order from 1 to 9.
(d) Its value when compared with the whole number.
(e) Its position in the number and its value.

10. In the number 4,632 the 3 represents a value how many times as large as the 6? _____
(a) 1/10 (b) 1/20 (c) 1/2 (d) 2 (e) 20

Basic Understandings of the Four Fundamental Processes with Whole Numbers

11. If you had a bag of 365 marbles to be shared equally by 5 boys, which would be the quickest way to determine each boy's share? _____
(a) dividing (b) counting (c) adding (d) subtracting
(e) multiplying

12. When a whole number is multiplied by a whole number other than 1, how does the answer compare with the whole number multiplied? _____
(a) larger (b) smaller (c) same (d) can't tell
(e) 10 times as large

13. When a whole number is divided by a whole number other than 1, how does the answer compare with the whole number divided? _____
(a) larger (b) smaller (c) same (d) can't tell
(e) 1/10 as large

14. Here is a subtraction example in which letters have been used instead of figures. Which of the following statements is true? _____
(a) ABCD and XYZ added together equal TRSM.
(b) XYZ and TRSM added together equal ABCD.
(c) ABCD and TRSM added together equal XYZ.
(d) TRSM subtracted from XYZ equal ABCD.
(e) XYZ subtracted from TRSM equals ABCD.

```
 ABCD
-XYZ
----
 TRSM
```

15. Adding two zeros to the right of a whole number has the same effect as: _____
(a) Adding ten to the number.
(b) Adding one hundred to the number.
(c) Multiplying the number by 10.
(d) Multiplying the number by 100.
(e) Dividing the number by 100.

16. How would the answer to this example be changed if you added two zeros to 439 and took away the zero from 450? _____
(a) The answer would be 10 times as large.
(b) The answer would be 100 times as large.
(c) The answer would stay the same.
(d) The answer would be 1/10 as large.
(e) The answer would be 1/100 as large.

```
 439
x450
----
```

17. Crossing off a zero from the right side of a number has the same effect as: _____
(a) subtracting 10 (b) subtracting 100 (c) multiplying by 10
(d) multiplying by 1 (e) dividing by 10

18. What would be the effect on the answer if you added two zeros to 38 and changed 6500 to 650? _____
(a) The answer would be 10 times as large.
(b) The answer would be 1/10 as large.
(c) The answer would be 100 times as large.
(d) The answer would be 1/100 as large.
(e) The answer would be 1/1000 as large.

38)6500

19. If the numbers in a long addition example were changed so that the top number was placed at the bottom and the bottom number was placed at the top, how would the answer be affected? _____
(a) answer would be larger (b) answer would be smaller
(c) answer would not change (d) could not do the example
(e) can't tell until you add both ways and compare.

20. How would this multiplication example be affected if you put the 47 above the 5648? _____
(a) The answer would be larger.
(b) The answer would be smaller.
(c) The answer would be the same.
(d) Can't tell until you multiply both ways and compare.
(e) You can't do the example when the large number is on the bottom and the small number on top.

```
 5648
x 47
-----
```

Basic Understandings of Fractions and the Four Fundamental Processes with Fractions.

21. Which of the following fractions is the largest? _____
(a) 1/9 (b) 7/9 (c) 5/9 (d) 11/9 (e) 8/9

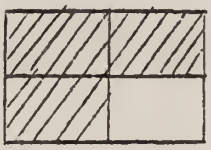
22. Which of these statements best tells why we can't say the unshaded parts of this picture represents 3 "eighths"? _____
(a) Because more than 5/8 of it is unshaded.
(b) Because the unshaded parts are not together.
(c) Because the unshaded parts are not the same size.
(d) Because less than 5/8 of it is unshaded.
(e) Because the parts are not the same shape.

23. Which of the following fractions is the smallest? _____
(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5 (e) 1/6

24. When a whole number is multiplied by a proper fraction, how does the answer compare with the whole number? _____
(a) larger (b) smaller (c) same (d) can't tell
(e) 1/2 as large

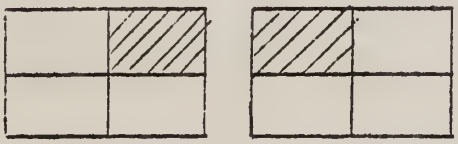
25. When a proper fraction is divided by a proper fraction, how does the answer compare with the fraction divided? _____
(a) larger (b) smaller (c) same (d) can't tell
(e) 2 times as large

26. Which picture shows how the result would look if you multiplied the denominator of this fraction by 2? _____



x 2

(a)

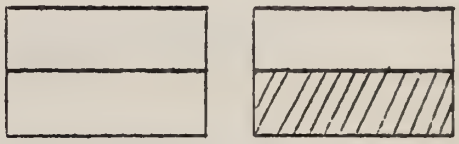


(b)

(c)

(d)

(e)



27. When a whole number is divided by a proper fraction, how does the answer compare with the whole number? _____
(a) larger (b) smaller (c) same (d) can't tell (e) varies

28. Which sentence best tells why the answer in this example is larger than five?

$5 \div 3/4 = 6\ 2/3$

(a) Because inverting the divisor turned the 3/4 upside down.
(b) Because multiplying always makes the answer larger.
(c) Because the divisor 3/4 is less than 1.
(d) Because dividing by proper and improper fractions makes the answer larger than the number divided.
(e) Because inverting a fraction puts the larger number on top.

29. Which of the following sentences are shown by this picture?

(a) Fractions with common denominators can be added.
(b) The value of a fraction is changed if a number is subtracted from the numerator and denominator.
(c) Dividing the numerator and denominator of a fraction by the same number does not change the value of the fraction.
(d) Fractions with the same denominators are equal.
(e) Fractions with the same numerators are equal.

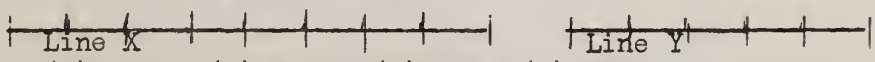
30. When a proper fraction is multiplied by a proper fraction how does the answer compare with the fraction multiplied?
(a) larger (b) smaller (c) same (d) can't tell (e) varies

Basic Understandings of Decimals and the Four Fundamental Processes with Decimals

31. How should you write the decimal eighty and eight hundredths?
(a) .8008 (b) 80.800 (c) 80.08 (d) 80.008 (e) 8008.08

32. How should you read this decimal - .0309?
(a) Three and nine thousandths
(b) Three hundred nine thousandths
(c) Three hundred nine ten thousandths
(d) Thirty-nine thousandths
(e) Three hundred nine hundredths

33. Which decimal tells how long <u>line Y</u> is when compared with <u>line X</u>?



(a) .5 (b) .625 (c) 1.25 (d) .75 (e) .33

34. What would be the effect on the answer if you dropped the zero from 23.90?
(a) The answer would have the same value.
(b) The answer would be 1/10 as large.
(c) The answer would be 10 as large.
(d) You would point off three places.
(e) It would be the same as subtracting zero from the answer.

$$\begin{array}{r} 23.90 \\ \underline{\times 2.75} \end{array}$$

35. Which seems to be the correct answer to this example:
ten divided by five tenths _____

(a) 1/2 (b) 2 (c) 10 (d) 20 (e) 50

36. Which of the following decimals has the largest value? _____
(a) 30.3 (b) 30.03 (c) 30.0333 (d) 30.303 (e) 30.003

37. What would be the effect on the answer if you changed 654 to 6540 and 38 to 3.8 in this example? _____

$$\begin{array}{r} 654 \\ \times\ 38 \\ \hline \end{array}$$

(a) The answer would be larger.
(b) The answer would be smaller.
(c) The answer remains the same.
(d) The answer would be 1/10 as large.
(e) Can't tell until you do example both ways.

38. How would the answer be affected if you moved the point one place to the left in both numbers of this example? _____

$$\begin{array}{r} 78.4 \\ \times 9.3 \\ \hline \end{array}$$

(a) The answer would be 1/10 as large.
(b) The answer would be 1/100 as large.
(c) The answer would be 100 times as large.
(d) It would be the same as subtracting 100 from the answer.
(e) The answer would have the same value.

39. How would the answer be affected if you moved the point in 724.9 one place to the right? _____

$27\overline{)724.9}$

(a) The answer would be 10 times as large.
(b) The answer would be 100 times larger.
(c) The answer would be 1/10 as large.
(d) The answer would have a zero at the left.
(e) The value of the answer would be the same.

40. Why is the answer smaller than the top number? _____

$$\begin{array}{r} 9 \\ \times .5 \\ \hline 4.5 \end{array}$$

(a) Because 9 is more than .5.
(b) Because you are finding how many .5's in 9.
(c) Because .5 is less than 9.
(d) When you multiply by a decimal the answer is always smaller than the top number.
(e) Because multiplying by .5 is the same as finding 1/2 the number.

Basic Understandings of Computations.

41. Why do we find a common denominator when adding fractions with unlike denominators? _____
(a) You can't add things that are different.
(b) It is easier to add fractions when they have a common denominator
(c) The denominators have to be the same in order to add.
(d) We learned to add unlike fractions that way.
(e) So that all fractions will have the same value.

42. When dividing by a decimal, why do we move the point to the right? _____
   (a) Multiplying by a multiple of 10 is a quick way of changing a decimal to a whole number.
   (b) It places the decimal point in the quotient correctly.
   (c) You can divide by a whole number.
   (d) To make the divisor equal the dividend.
   (e) It is easier to divide by a whole number than a decimal.

43. Which one of the following would give the correct answer to the example? 2.1 x 21 _____

   (a) The sum of 1 x 2.1 and 21 x 2.1
   (b) The sum of 10 x 2.1 and 2 x 2.1
   (c) The sum of 10 x 2.1 and 20 x 2.1
   (d) The sum of 1 x 2.1 and 20 x 2.1
   (e) The sum of 1 x 2.1 and 2 x 2.1

44. Which statement best tells why we invert the divisor and multiply when dividing a fraction by a fraction? _____
   (a) It is an easy method of finding a common denominator and arranging the numerators in multiplication form.
   (b) It is an easy method for dividing the denominators and multiplying the numerators of the two fractions.
   (c) It is a quick way of arranging two fractions in multiplication form.
   (d) Dividing a fraction is the same as multiplying by the reciprocal of the fraction.
   (e) It is a quick method of finding the reciprocals of both fractions and reducing to lowest terms.

45. Why do we move the second partial product one place to the left when we multiply by the 6? _____

```
  579
 x 69
```

   (a) Because the answer has to be larger than 579.
   (b) Because the six means six tens.
   (c) Because the six is the second figure in 69.
   (d) Because we learned to multiply that way.
   (e) Because the six represents a larger value than the nine.

46. Which statement best tells why we arrange numbers in addition in columns the way we do? _____
   (a) It is easier to keep the numbers in straight columns.
   (b) It helps to add correctly.
   (c) It helps us add only those numbers in the same position.
   (d) It helps us to carry correctly from one column to another.
   (e) It would be harder to add if the numbers were mixed.

47. In this example you multiply by the 6 then by the 3. How do the two results (partial products) compare? _____

```
 974
x36
```

   (a) The second product is one-half the first.
   (b) The second product is twice as large as the first.
   (c) The second product is five times the first.
   (d) The second product is ten times the first.
   (e) The second product is 20 times the first.

48. Which statement best explains the 4 in the answer?

```
        48
26) 1248
    104
     208
     208
```

(a) The 4 means there are 48 26's in 1248.
(b) The 4 means there are 4 26's in 1248.
(c) The 4 means that 26 goes into 124 4 times and 5 is too large.
(d) The 4 means that there are at least 40 26's in 1248.
(e) The 4 means that the answer will come out even.

49. Here is an example in subtraction of mixed numbers in which it is necessary to borrow. Which statement best explains the borrowing?

```
  5 3/8
—2 5/8
```

(a) You can't subtract 5/8 from 3/8 so you take 1 from the 5 and put it in front of the 3 making 13.
(b) You can't subtract 5/8 from 3/8 so you add the 3 and 8 making 11/8.
(c) You can't subtract 5/8 from 3/8 so you take 1 from the 5 and add it to 3/8 making it 4/8.
(d) You can't subtract 5/8 from 3/8 so you take 1 from the 5 making it 4 and add the 1 to the 3/8 making it 4/8.
(e) You can't subtract 5/8 from 3/8 so you take 1 from the 5 and change it to 8/8, then add the 8/8 to 3/8 making 11/8.

50. Which statement best explains what happens when you reduce a fraction to lowest terms?

(a) The size of the terms and the value of the fraction become smaller.
(b) The value of the fraction does not change. The size of the part represented by the new denominator is smaller and the number of parts represented by the new numerator is less.
(c) The value of the fraction does not change. The terms are smaller.
(d) The value of the fraction does not change, but the parts of the fraction represented by the new numbers become fewer in number and smaller in size.
(e) The value of the fraction changed because the new numbers are smaller.

Catalog No. BF-2507
Manufactured by
National Photographic Envelope Co.
Phila., Pa.